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A Borel–Carathéodory Inequality and Approximation of Entire Functions by Polynomials with Restricted Zeros*

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1. INTRODUCTION

By the Borel–Carathéodory inequality, if f is holomorphic in $|z| < 1$ with $\operatorname{Re} f(z) \leqslant A$ $(A > 0)$, then

$$\|f(z)\|_{|z|\leqslant r<1} \leqslant \frac{1+r}{1-r}(A + |f(0)|), \tag{1.1}$$

where the left hand side denotes the sup norm of f over the closed disc $|z| \leqslant r < 1$. We make use of this result to prove a similar inequality:

THEOREM 1. *Let γ be a closed, nondegenerate arc of $|z| = 1$. Let f be holomorphic in $|z| < 1$ with $\operatorname{Re} f(z) < A$ $(A > 0)$ there and continuous in $\{|z| < 1\} \cup \gamma$ with $\|f\|_\gamma \leqslant M$. For any compact subset S of $\gamma^0 \cup \{|z| < 1\}$ there is a constant C_S, dependent only on S, such that*

$$\|f\|_S \leqslant C_S(A + M). \tag{1.2}$$

(γ^0 *is* γ *with its end points removed.*)

We use Theorem 1 to generalize a classical Theorem of Laguerre–Pólya [6, Theorem XII]: if D is an open half-plane and $\{P_n(z)\}$ is a sequence of D^C-polynomials (polynomials whose zeros lie in the complement of D) which converges uniformly on a disc in D to some $f \not\equiv 0$, then $\{P_n(z)\}$ converges uniformly on every compact set and f is entire of order $\leqslant 2$. This result was generalized in one direction by Ganelius [2] who extended it to simply con-

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nected domains containing a half-plane. Later Korevaar and Loewner [5] proved that when D is an open half-plane, it is sufficient to have uniform convergence to $f \not\equiv 0$ on a Jordan arc in clos D. These results follow as special cases of:

THEOREM 2. *Let D be a simply connected domain containing a half-plane. Let γ be a Jordan arc in* clos D *such that either $\gamma \subset D$ or γ is a free Jordan arc for D (that is, γ^0 contains only accessible boundary points of D and every Jordan domain bounded by a subarc of γ^0 and two cuts into D lies entirely in D). If $\{P_n(z)\}$ is a sequence of D^C-polynomials which converges uniformly on γ to some $f \not\equiv 0$, then $\{P_n(z)\}$ converges uniformly on every compact set and f is entire of order $\leqslant 2$. (For a more precise description of the form of f see* [6].)

2. THE VARIANT OF THE BOREL–CARATHÉODORY INEQUALITY

Theorem 1 follows quite easily from Lemma 2.1, whose proof was suggested by F. Carroll.

LEMMA 2.1. *Let $\gamma = \{e^{i\theta} : |\theta| \leqslant \theta_0 < \pi\}$. Suppose f is holomorphic in $|z| < 1$ with* $\operatorname{Re} f(z) > 0$ *there and continuous on $\{|z| < 1\} \cup \gamma$ with $\|f\|_\gamma \leqslant 1$. For each closed sector $S = \{z : |\arg z| \leqslant \theta_1 < \theta_0, |z| \leqslant 1\}$ there is a constant C_S dependent only on S, such that $\|f\|_S \leqslant C_S$* $\max(1, |f(0)|)$.

Proof. Let $z_1 = \exp(i(\theta_0 + \theta_1)/2)$ and $z_2 = \bar{z}_1$. We define:

$$h(z) = (z - z_1)(z - z_2) f(z). \tag{2.1}$$

Then,

$$\|h\|_\gamma \leqslant 4 \|f\|_\gamma \leqslant 4, \tag{2.2}$$

and by applying the Borel–Carathéodory inequality to $f(z)$, we obtain

$$|h(rz_1)| \leqslant 2(1 - r) \frac{(1 + r)}{(1 - r)} |f(0)| \leqslant 4 |f(0)|. \tag{2.3}$$

Similarly,

$$|h(rz_2)| \leqslant 4 |f(0)|. \tag{2.4}$$

But $h(z)$ is holomorphic in $|z| < 1$, so that by (2.2), (2.3) and (2.4):

$$\|h\|_S \leqslant 4 \max(|f(0)|, 1). \tag{2.5}$$

It follows from (2.1) and (2.5) that

$$\|f\|_S \leqslant \frac{\|h\|_S}{|e^{i\theta_1} - z_1|^2} \leqslant \frac{4}{|e^{i\theta_1} - z_1|^2} \max(1, |f(0)|). \tag{2.6}$$

LEMMA 2.2. *Let* γ, f *and* S *be as in Lemma* 2.1 *and let*

$$U = \{|z| \leqslant r_1 < 1\} \cup S.$$

There exists a constant C_U, *dependent only on* U, *such that*

$$\|f\|_U \leqslant C_U .$$

Proof. By Lemma 2.1, $\|f\|_S \leqslant C_S \max(1, |f(0)|)$. By the Borel–Carathéodory inequality, $\|f\|_{|z| \leqslant r_1} \leqslant [(1 + r_1)/(1 - r_1)] \max(1, |f(0)|)$. Thus $\|f\|_U \leqslant K_U \max(1, |f(0)|)$, where $K_U = C_S + [(1 + r_1)/(1 - r_1)] > 1$.

If $|f(0)| \leqslant 1$, then

$$\|f\|_U \leqslant K_U . \tag{2.7}$$

If $|f(0)| > 1$, then $\|f\|_U \leqslant K_U |f(0)|$ and by the two constants theorem [3, Vol. 2, p. 409],

$$|f(0)| \leqslant (K_U |f(0)|)^\lambda \cdot 1^{1-\lambda},$$

where λ $(0 < \lambda < 1)$ is the harmonic measure of $\{e^{i\theta} : |\theta| \leqslant \theta_1\}$ with respect to U^0, the interior of U, evaluated at 0. So $|f(0)| \leqslant K_U^{\lambda/(1-\lambda)}$ and

$$\|f\|_U \leqslant K_U^{1/(1-\lambda)}. \tag{2.8}$$

By (2.7) and (2.8) we always have $\|f\|_U \leqslant K_U^{1/(1-\lambda)}$.

Proof of Theorem 1. Without loss of generality, we may assume

$$\gamma = \{e^{i\theta} : |\theta| \leqslant \theta_0 < \pi\}$$

and

$$S = \{z : |z| \leqslant r_1 < 1\} \cup \{z : |\arg z| \leqslant \theta_1 < \theta_0 , |z| \leqslant 1\}.$$

Let $g(z) = (A - f(z))/(A + M)$. By Lemma 2.2, $\|g\|_S \leqslant C_S$, so that $\|f\|_S \leqslant (1 + C_S)(A + M)$.

3. Generalization of the Laguerre-Pólya's Theorem

Without loss of generality, we may state Theorem 2 in the following form:

THEOREM 3. *Let D be a simply-connected domain containing* $\{\operatorname{Re} z > a\}$. *Let γ be a free Jordan arc for D with $0 \in \gamma^0$. If $\{P_n(z) = \prod_{K=1}^{\nu_n} (1 - (z/z_K^{(n)}))\}$ is a sequence of D^C-polynomials which converges uniformly to some $f \not\equiv 0$ on γ, then $\{P_n(z)\}$ converges uniformly on every compact set and the limit function is entire of order $\leqslant 2$.*

Proof. We may assume

$$\| P_n(z) - 1 \|_\gamma \leqslant \tfrac{1}{2}. \tag{3.1}$$

Let E be a closed disc of radius δ about z_0 contained in $\{\operatorname{Re} z > a\}$. There is a bounded simply connected subdomain S of D such that $S \supset E$ and $\gamma' = \partial S \cap \gamma$ is a free Jordan arc for S with $0 \in \gamma'$. Let $\log P_n(z)$ denote a continuous branch of the logarithm on $S \cup \gamma'$ such that $\log P_n(0) = 0$.

Following the method of Ganelius [2], we have for all z

$$\log \left| \frac{P_n(z + z_0)}{P_n(z_0)} \right| \leqslant \left(| z | + \frac{C}{\delta} | z |^2 \right) \frac{M_n}{\delta}, \tag{3.2}$$

where C is independent of z and n, and where $M_n = \max_{z \in E} | \log P_n(z)|$. If T is any bounded set, we have by (3.2),

$$\sup_{z \in T} \log | P_n(z)| \leqslant K_T M_n, \tag{3.3}$$

where K_T is independent of n.

Let γ'' be a (proper) subarc of $(\gamma')^0$. There is a closed Jordan region $S_1 \subset S \cup \gamma'$ such that S_1^0 contains E and $\gamma'' = \partial S_1 \cap \gamma'$. Since γ' is a free Jordan arc for S, any conformal mapping of $| w | < 1$ onto S will have a one-to-one continuous extension from an arc C_w of $| w | = 1$ onto γ' [1, p. 86]. Thus by (3.1) and (3.3), we can apply Theorem 1 to $\log P_n(z)$ and obtain

$$\| \log P_n \|_{S_1} \leqslant D_{S_1}(K_S M_n + 1), \tag{3.4}$$

where D_{S_1} depends only on S_1 and not on n.

Either $M_n < 1/K_S$ or $M_n \geqslant 1/K_S$. In the latter case, there is a λ independent of n such that $0 < \lambda < 1$, and

$$M_n \leqslant \| \log P_n \|_\gamma^\lambda \cdot \| \log P_n \|_{S_1}^{1-\lambda}, \tag{3.5}$$

by the two constants theorem. Combining (3.4) and (3.5),

$$\| \log P_n \|_{S_1} \leqslant 2D_{S_1}K_S \| \log P_n \|_\gamma^\lambda \cdot \| \log P_n \|_{S_1}^{1-\lambda},$$

so that by (3.1),

$$\| \log P_n \|_{S_1} \leqslant (2D_{S_1}K_S)^{1/\lambda} \| \log P_n \|_\gamma \leqslant (2D_{S_1}K_S)^{1/\lambda}.$$

Thus for all n,

$$M_n \leqslant \frac{1}{K_S} + (2D_{S_1}K_S)^{1/\lambda} = M, \tag{3.6}$$

and by (3.3) and (3.6), for any bounded set T,

$$\sup_{z \in T} | P_n(z)| \leqslant e^{K_T M}.$$

It now follows that $\{P_n(z)\}$ is normal on $\{| z | < \infty\}$ and since $\{P_n(z)\}$ converges uniformly on γ to $f \not\equiv 0$, the sequence converges uniformly on every compact set to some $g \not\equiv 0$. By the Laguerre-Pólya result, g is entire of order $\leqslant 2$.

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